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A derivation of the Nambu–Goto action from invariance principles

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Abstract. We show that the Nambu–Goto action of string theory can be obtained from the invariance with respect to the reparametrization of the world sheet and the invariance with respect to the Poincaré transformations. The proof is based on the geometrical Lagrangian approach of Souriau. In a three-dimensional Minkowski space a Chern–Simons term can appear.

1. Introduction

The Nambu–Goto action for the relativistic string is usually postulated by analogy with the relativistic action for a point-like particle (see for instance [1]). On the other hand, it is known that the relativistic action for the point-like particle can be derived rigorously using only the Poincaré invariance [2, 3]. So, it is natural to expect that a similar statement is valid for the Nambu–Goto action as well. The purpose of this paper is to prove this point.

In section 2 we will present the general formalism. This formalism uses only the Lagrangian density and is based on geometrical ideas originating in the work of Lagrange, used systematically in particle mechanics by Souriau [4] and generalized to dynamical systems with an infinite number of degrees of freedom in [5–7] (see also [8]). Recently this formalism was used in the analysis of gauge invariance [9]. Here we use the same techniques to analyse the general case of a p-brane postulating the invariance with respect to reparametrizations of the world sheet and the usual Poincaré invariance. Some comments regarding these invariance postulates are also made in this section.

In section 3 we analyse the general conditions obtained in section 2 in the simplest case $p = 1$, i.e. the relativistic particle. We derive the usual expression for the Lagrangian (of the homogeneous formulation). We have included the analysis of this case for two reasons. First, the proof is different from the proofs in [2, 3] and does not use cohomology arguments as in [2]. Second, it is rather simple and illustrates the general ideas, so it is a good guide for the more complicated case $p > 1$. Let us mention that an analysis using similar ideas was done in [10].

In section 4 we analyse the case $p = 2$, i.e. the relativistic string. We are able to prove that the Poincaré invariance and the reparametrization invariance fix up to a multiplicative constant, the Lagrangian density in Minkowski spaces of dimension different from 3; namely, we obtain the Nambu–Goto Lagrangian. In a three-

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dimensional Minkowski space the Lagrangian is, in general, a sum of the Nambu–Goto term and a term of a Chern–Simons nature. Terms of this type have already appeared in the literature (see for instance [11] formula (3.4.46)) but they are more general and have been postulated from different considerations. Our analysis shows that there exists essentially a single expression of this type compatible with Poincaré and reparametrization invariance.

The case $p > 2$ is not analysed here because of computational complexity.

2. The consequences of the reparametrization and of the Poincaré invariance for a p-brane

2.1. *The general formalism*

We present here the general formalism. Let S be a differentiable manifold of dimension $p + D$. To build a first-order Lagrangian formalism we need an auxiliary object, namely the bundle of 1-jets of p-dimensional submanifolds of S, denoted $J^1_p(S)$. This manifold is by definition:

$$J^1_p(S) \equiv \cup_{s\in S} J^1_p(S)_s$$

where $J^1_p(S)_s$ is the manifold of p-dimensional linear subspaces of the tangent space $T_s(S)$ at S in the point $s \in S$. This manifold is naturally fibred over S and we denote the canonical projection by π.

Let us choose a coordinate system (x^μ, ψ^A) on the open set $U \subseteq S$; here $\mu = 1, \ldots, p$ and $A = 1, \ldots, D$. Then, on the open set $V \subseteq \pi^{-1}(U)$ we shall choose the coordinate system $(x^\mu, \psi^A, \chi^A_\mu)$ where by definition, the p-plane in $T_{s_0}(S)$ corresponding to the set of numbers $(x^\mu_0, \psi^A_0, (\chi^A_\mu)_0)$ (here (x^μ_0, ψ^A_0) are the coordinates of $s_0 \in U$) is spanned by the tangent vectors

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + (\chi^A_\mu)_0 \frac{\partial}{\partial \psi^A}. \tag{2.1}$$

We will systematically use the convention of summation over the dummy indices.

The Lagrangian formalism describes p-dimensional immersed submanifolds usually given in a parametrized form $\Psi : P \to S$ (P is a p-dimensional manifold). We denote by $\dot\Psi : P \to J^1_p(S)$ the natural lift of Ψ. In the local coordinates shown earlier, if Ψ is given by $x^\mu \mapsto (x^\mu, \psi^A(x))$, then $\dot\Psi$ is given by $x^\mu \mapsto (x^\mu, \psi^A(x), (\partial\psi^A/\partial x^\mu)(x))$. A (local) *Lagrangian function* (or *density*) is a smooth real function L defined on a subset $J^1_p(S)$. Then the *Euler–Lagrange equations* for Ψ can be written in the local coordinates above as follows

$$E_A(\Psi) \equiv \frac{\partial L}{\partial \psi^A} \circ \dot\Psi - \frac{\partial}{\partial x^\mu}\left[\frac{\partial L}{\partial \chi^A_\mu} \circ \dot\Psi\right] = 0. \tag{2.2}$$

We define now the Poincaré–Cartan form associated with the local Lagrangian L to be the p-form θ_L given in local coordinates by

$$\theta_L \equiv \varepsilon_{\mu_1,\ldots,\mu_p} \sum_{k=0}^{p} \frac{1}{k!}\binom{p}{k} L^{\mu_1,\ldots,\mu_k}_{A_1,\ldots,A_k} \delta\psi^{A_1} \wedge \ldots \wedge \delta\psi^{A_k} \wedge \mathrm{d}x^{\mu_{k+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}. \tag{2.3}$$

Here $\varepsilon_{\mu_1,\ldots,\mu_p}$ is the signature of the permutation $(1,\ldots,p) \mapsto (\mu_1,\ldots,\mu_p)$, $\delta\psi^A$ is, by definition,

$$\delta\psi^A \equiv \mathrm{d}\psi^A - \chi^A_\mu \mathrm{d}x^\mu \tag{2.4}$$

and $L^{\mu_1,\ldots,\mu_k}_{A_1,\ldots,A_k}$ is given by

$$L^{\mu_1,\ldots,\mu_k}_{A_1,\ldots,A_k} \equiv \frac{1}{k!}\sum_{\sigma\in P_k}(-1)^{|\sigma|}\frac{\partial^k L}{\partial\chi^{A_1}{}_{\mu_{\sigma(1)}}\ldots\partial\chi^{A_k}{}_{\mu_{\sigma(k)}}}. \tag{2.5}$$

(P_k is the group of the permutation of the numbers $1,\ldots,k$ and $|\sigma|$ is the signature of σ.) Note that $L^{\mu_1,\ldots,\mu_k}_{A_1,\ldots,A_k}$ is completely antisymmetric in the upper indices and also in the lower indices.

The Lagrange–Souriau form σ_L is then defined locally by

$$\sigma_L \equiv \mathrm{d}\theta_L. \tag{2.6}$$

This $(p+1)$-form has some remarkable properties [5–7]:

(P1) σ_L given by (2.6) can be globally defined; more precisely, one can define a global $(p+1)$-form σ such that in any chart with local coordinates as above one can find a local Lagrangian L having the property $\sigma = \sigma_L$ [8].

(P2) The p-dimensional immersed submanifold $\Psi : P \to S$ satisfies the Euler–Lagrange equations (2.2) iff

$$(\dot\Psi)^*(\mathrm{i}_Z\sigma_L) = 0$$

for any vector field Z on $J^1_n(S)$.

(P3) A Lagrangian L gives trivial Euler–Lagrange equations iff

$$\sigma_L = 0.$$

(P4) A transformation $\Phi \in \mathrm{Diff}(J^1_p(S))$ is a *Noetherian symmetry* for L (i.e. leaves the action functional unchanged up to a trivial action) iff

$$\Phi^*\sigma_L = \sigma_L. \tag{2.7}$$

By a trivial action we mean an action giving trivial Euler–Lagrange equations.

It is this last property which will be the key to our proof. Let us note also that usually one considers only *restricted Noetherian symmetries*, i.e. maps Φ which are of the form: $\Phi = \dot\phi$, where $\phi \in \mathrm{Diff}(S)$ and $\dot\phi \in \mathrm{Diff}(J^1_p(S))$ is the natural lift of ϕ to $J^1_p(S)$.

Finally, for practical computations, we need an explicit expression for σ_L. By differentiating (2.3) and rearranging the terms, one gets after some computations the following expression:

$$\sigma_L = \varepsilon_{\mu_1,\ldots,\mu_p}\sum_{k=0}^{p}\frac{1}{k!}\binom{p}{k}\Big[\sigma^{\mu_0,\ldots,\mu_k}_{A_0,\ldots,A_k}\mathrm{d}\chi^{A_0}_{\mu_0}\wedge\delta\psi^{A_1}\ldots\wedge\delta\psi^{A_k}\wedge\mathrm{d}x^{\mu_{k+1}}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}$$
$$+\rho^{\mu_1,\ldots,\mu_k}_{A_0,\ldots,A_k}\delta\psi^{A_0}\wedge\ldots\wedge\delta\psi^{A_k}\wedge dx^{\mu_{k+1}}\wedge\ldots\wedge\mathrm{d}x^{\mu_p}\Big] \tag{2.8}$$

where

$$\sigma^{\mu_0,\ldots,\mu_k}_{A_0,\ldots,A_k} \equiv \frac{\partial L^{\mu_1,\ldots,\mu_k}_{A_1,\ldots,A_k}}{\partial\chi^{A_0}_{\mu_0}} - L^{\mu_0,\ldots,\mu_k}_{A_0,\ldots,A_k} \tag{2.9}$$

and

$$\rho^{\mu_1,\ldots,\mu_k}_{A_0,\ldots,A_k} \equiv \frac{1}{k+1}\left(\sum_{i=0}^{k}(-1)^i\frac{\partial L^{\mu_1,\ldots\ldots\ldots,\mu_k}_{A_0,\ldots,\hat{A}_i,\ldots,A_k}}{\partial\psi^{A_i}} - \frac{\delta L^{\mu_0,\ldots,\mu_k}_{A_0,\ldots,A_k}}{\delta x^{\mu_0}}\right). \tag{2.10}$$

Here a hat over an index means, as usual, an omission.

2.2. *The geometric formalism for a p-brane*

We particularize the general framework for the case of a p-brane. We take, in the general scheme from section 2.1, $S = P \times M$. Here P is the manifold of the parameters characterizing our extended object, with coordinates usually denoted by τ^a; $a = 1,\ldots,p$. For instance, for a point-like particle $p = 1$ and τ parametrize the world line. For a string, $p = 2$ and the parameters (τ^1, τ^2) describe the world sheet. If the string is open, then we can take $P = R^2$, and if the string is closed, we can take $P = R \times S^1$. In general P can be an arbitrary Riemann manifold. Up to a point, the analysis is somewhat insensitive to the specific choice of P. We have denoted by M the D-dimensional Minkowski space, identified as usual with $\mathbb{R}^D$ and with coordinates denoted by X^μ; $\mu = 1,\ldots,D$. The coordinates on $J^1_p(S)$ are $(\tau^a, X^\mu, U^\mu{}_a)$. We now particularize (2.1)–(2.10). We have

$$\frac{\delta}{\delta\tau^a} \equiv \frac{\partial}{\partial\tau^a} + U^\mu{}_a \frac{\partial}{\partial X^\mu} \tag{2.11}$$

$$\delta X^\mu \equiv \mathrm{d}X^\mu - U^\mu{}_a \mathrm{d}\tau^a \tag{2.12}$$

$$\begin{aligned}\sigma_L = \varepsilon_{\mu_1,\ldots,\mu_p} \sum_{k=0}^{p} \frac{1}{k!}\binom{p}{k} [\sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \mathrm{d}U^{\mu_0}{}_{a_0} \wedge \delta X^{\mu_1} \wedge\ldots\wedge \delta X^{\mu_k} \wedge \mathrm{d}\tau^{a_{k+1}} \wedge\ldots\wedge \mathrm{d}\tau^{a_p} \\ + \rho^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \delta X^{\mu_0} \wedge\ldots\wedge \delta X^{\mu_k} \wedge \mathrm{d}\tau^{a_{k+1}} \wedge\ldots\wedge \mathrm{d}\tau^{a_p}\end{aligned} \tag{2.13}$$

where

$$\sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \equiv \frac{\partial L^{a_1,\ldots,a_k}_{\mu_1,\ldots,\mu_k}}{\partial U^{\mu_0}{}_{a_0}} - L^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \tag{2.14}$$

$$\rho^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \equiv \frac{1}{k+1}\left(\sum_{i=0}^{k} (-1)^i \frac{\partial L^{a_1,\ldots\ldots\ldots,a_k}_{\mu_0,\ldots,\hat{\mu}_i,\ldots,\mu_k}}{\partial X^{\mu_i}} - \frac{\delta L^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k}}{\delta\tau^{a_0}} \right) \tag{2.15}$$

and

$$L^{a_1,\ldots a_k}_{\mu_1,\ldots,\mu_k} \equiv \frac{1}{k!} \sum_{\sigma\in P_k} (-1)^{|\sigma|} \frac{\partial^k L}{\partial U^{\nu_1}{}_{\mu_{\sigma(1)}} \ldots \partial U^{\nu_k}{}_{\mu_{\sigma(k)}}}. \tag{2.16}$$

2.3. *The reparametrization invariance*

Next, we analyse the reparametrization invariance. Let $\xi \in \mathrm{Diff}(P)$. Then, the action of a reparametrization transformation on the manifold S is simply

$$\phi_\xi(\tau, X) = (\xi(\tau), X). \tag{2.17}$$

One computes without difficulty the lift of this map to $J^1_p(S)$ and gets

$$\dot{\phi}_\xi(\tau, X, U^\mu{}_a) = (\xi(\tau), X, A_a{}^b(\tau) U^\mu{}_b). \tag{2.18}$$

Here $A_a{}^b(\tau)$ is the inverse of the $p \times p$ matrix $(\partial\xi^a/\partial\tau^b)$, i.e.

$$\frac{\partial\xi^b}{\partial\tau^a}(\tau) A_b{}^c(\tau) = \delta^c_a. \tag{2.19}$$

Then it is clear that the theory is reparametrization invariant iff $\dot{\phi}_\xi$ is a Noetherian symmetry, i.e.

$$(\dot{\phi}_\xi)^* \sigma_L = \sigma_L. \tag{2.20}$$

for $\forall \xi \in \mathrm{Diff}(P)$.

It is not hard to prove this condition is equivalent to

$$A_{a_0}{}^{b_0} \ldots A_{a_k}{}^{b_k} \sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \circ \dot{\phi}_\xi = (\det(A)) \sigma^{b_0,\ldots,b_k}_{\mu_0,\ldots,\mu_k} \tag{2.21}$$

and

$$A_{a_1}{}^{b_1} \ldots A_{a_k}{}^{b_k} \left(\rho^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \circ \dot{\phi}_\xi - \frac{1}{k+1} \frac{\partial A_b{}^d}{\partial \tau^c} U^\nu{}_d \sigma^{bca_1,\ldots,a_k}_{\nu\mu_0,\ldots,\mu_k} \circ \dot{\phi}_\xi \right)$$
$$= (\det(A)) \rho^{b_1,\ldots,b_k}_{\mu_0,\ldots,\mu_k}. \tag{2.22}$$

Let us comment on the physical significance of the reparametrization invariance. First, we remind the reader that a symmetry of the Euler–Lagrange equation of motion is a diffeomorphism of $J^1_p(S)$ mapping solutions of these equations into solutions of the same equations. Noetherian symmetries are just a particular case. So, reparametrization invariance means in our case, that if Ψ is a solution of (2.2) then by reparametrizing it we obtain a new solution $\Psi \circ \xi$ of the same equations (2.2).

This translates physically in the geometric character of the theory i.e. the parameters τ have no physical meaning and can be changed at will by a reparametrization: in another words they are just some labels.

In the following we will consider in (2.20) only the diffeomorphisms ξ belonging to the connected component of the identity in $\mathrm{Diff}(P)$, i.e. $\xi \in (\mathrm{Diff}(P))_0$. If $\xi \in (\mathrm{Diff}(P))_0$ then without losing any information, we can consider in (2.21) and (2.22) that ξ is an infinitesimal transformation, i.e.

$$\xi^a(\tau) = \tau^a + \theta^a(\tau) \tag{2.23}$$

with θ infinitesimally small. We insert (2.23) into (2.21) and (2.22) and we keep only the first-order terms in θ. Taking into account that θ is an arbitrary function, it is straightforward to establish that $\sigma^{\cdots}_{\cdots}$ and $\rho^{\cdots}_{\cdots}$ do not depend on τ and also that they obey the following relations:

$$U^\nu{}_b \frac{\partial \sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k}}{\partial U^\nu{}_c} = \delta^c_b \sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k} - \sum_{i=0}^{k} \delta^{a_i}_b \sigma^{a_0,\ldots,a_{i-1},c,a_{i+1},\ldots,a_k}_{\mu_0,\ldots\ldots\ldots\ldots,\mu_k} \tag{2.24}$$

$$U^\nu{}_d \sigma^{bca_1,\ldots,a_k}_{\nu\mu_0,\ldots,\mu_k} + (b \leftrightarrow c) = 0 \tag{2.25}$$

$$U^\nu{}_b \frac{\partial \rho^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k}}{\partial U^\nu{}_c} = \delta^c_b \rho^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} - \sum_{i=1}^{k} \delta^{a_i}_b \rho^{a_1,\ldots,a_{i-1},c,a_{i+1},\ldots,a_k}_{\mu_0,\ldots\ldots\ldots\ldots,\mu_k}. \tag{2.26}$$

2.4. *The Poincaré invariance*

The proper orthochronous Poincaré transformations act on S as follows

$$\phi_{L,a}(\tau, X) = (\tau, LX + a). \tag{2.27}$$

Here $a \in M \cong \mathbb{R}^D$ and $L \in \mathrm{End}(M)$ is a proper orthochronous Lorentz transformation. An elementary computation yields:

$$\dot{\phi}_{L,a}(\tau, X, U) = (\tau, LX + a, LU) \tag{2.28}$$

where

$$(LU)^{\mu}{}_{a} \equiv L^{\mu}{}_{\nu} U^{\nu}{}_{a}. \tag{2.29}$$

Then the condition of Poincaré invariance is simply

$$(\dot{\phi}_{L,a})^{*} \sigma_L = \sigma_L. \tag{2.30}$$

It is elementary to prove that (2.30) is equivalent to the fact that the functions $\sigma^{\cdots}_{\cdots}$ and $\rho^{\cdots}_{\cdots}$ do not depend on X and also they obey the following relations:

$$L^{\mu_0}{}_{\nu_0} \ldots L^{\mu_k}{}_{\nu_k} \sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \circ \dot{\phi}_{L,0} = \sigma^{a_0,\ldots,a_k}_{\nu_0,\ldots,\nu_k} \tag{2.31}$$

$$L^{\mu_0}{}_{\nu_0} \ldots L^{\mu_k}{}_{\nu_k} \rho^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \circ \dot{\phi}_{L,0} = \rho^{a_1,\ldots,a_k}_{\nu_0,\ldots,\nu_k} \tag{2.32}$$

for any proper orthochronous Lorentz transformation L, i.e. they are Lorentz covariant tensor functions.

2.5. *The line of computation*

So, finally we get that the reparametrization and the Poincaré invariance are equivalent to the fact that the functions $\sigma^{\cdots}_{\cdots}$ and $\rho^{\cdots}_{\cdots}$ depend only on U and verify the relations (2.24)–(2.26) and (2.31)–(2.32). After determining the most general form of $\sigma^{\cdots}_{\cdots}$ and $\rho^{\cdots}_{\cdots}$ verifying these equations, we use (2.14)–(2.16) to determine the most general form of L. This programme is feasible in principle for any p, but in practice we have only succeeded in performing it for $p = 1$ and $p = 2$. These two cases will be treated in detail in the following two sections.

3. The relativistic point-like particle

3.1. *The relevant formulae*

First, we particularize the formulae (2.14) and (2.15) to the case $p = 1$. From (2.13) it is clear that in this case we have only three functions to analyse: $\sigma_{\mu\nu}, \rho_{\mu}$ and $\rho_{\mu\nu}$. (Because the indices $a, b, \ldots$ can take only one value we have suppressed them.) From (2.14) we have, for $k = 1$,

$$\sigma_{\mu\nu} = \frac{\partial^2 L}{\partial U^{\mu} \partial U^{\nu}} \tag{3.1}$$

and from (2.15) for $k = 0$ and $k = 1$, respectively,

$$\rho_{\mu} = \frac{\partial L}{\partial X^{\mu}} - \frac{\delta L_{\mu}}{\delta \tau} \tag{3.2}$$

$$\rho_{\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 L}{\partial X^{\mu} \partial U^{\nu}} - (\mu \leftrightarrow \nu)\right). \tag{3.3}$$

3.2. *The functions* $\sigma_{\mu\nu}$

We first determine $\sigma_{\mu\nu}$. From the Lorentz covariance (2.31) and the symmetry in μ and ν (see (3.1)) it follows that $\sigma_{\mu\nu}$ is of the following form:

$$\sigma_{\mu\nu}(U) = G_{\mu\nu} f(\|U\|^2) + U^\mu U^\nu g(\|U\|^2). \tag{3.4}$$

Here $G_{\mu\nu}$ is the Minkowski bilinear form on M, f and g are smooth functions of the variable

$$\zeta \equiv \|U\|^2 \equiv G_{\mu\nu} U^\mu U^\nu \tag{3.5}$$

and we have taken into account that, according to section 2.5, $\sigma_{\mu\nu}$ is a only function of U.

Now we use the reparametrization invariance as expressed by the formulae (2.24) and (2.25). In our case these two relations became

$$U^\omega \frac{\partial \sigma_{\mu\nu}}{\partial U^\omega} = -\sigma_{\mu\nu} \tag{3.6}$$

$$U^\mu \sigma_{\mu\nu} = 0. \tag{3.7}$$

Inserting (3.4) into (3.7) we easily get that

$$f(\zeta) = -\zeta g(\zeta)$$

i.e. instead of (3.4) we can take

$$\sigma_{\mu\nu}(U) = g(\|U\|^2)(U_\mu U_\nu - G_{\mu\nu}\|U\|^2). \tag{3.8}$$

Inserting this relation into (3.6) we obtain a differential equation for the function g:

$$2\zeta g'(\zeta) + 3g(\zeta) = 0$$

which implies:

$$\zeta^3[g(\zeta)]^2 = \text{constant}. \tag{3.9}$$

We have three possibilities:

(a) constant > 0. In this case L must be defined only on the domain

$$D_+ \equiv \{(\tau, X, U) \,|\, \|U\|^2 > 0\}$$

then (3.9) gives

$$g(\zeta) = m/\zeta^{3/2} \tag{3.10}$$

for some $m \in \mathbb{R}$.

(b) constant < 0. In this case L must be defined on the domain

$$D_- \equiv \{(\tau, X, U) \,|\, \|U\|^2 < 0\}$$

and (3.9) gives

$$g(\zeta) = m/(-\zeta)^{3/2} \tag{3.11}$$

for some $m \in \mathbb{R}$.

(c) constant $= 0$. The Lagrangian can be defined everywhere and (3.9) gives

$$g = 0. \tag{3.12}$$

3.3. *The Lagrangian*

We analyse these three cases separately.

(a) From (3.1) and (3.10) we have

$$\frac{\partial^2 L}{\partial U^\mu \partial U^\nu} = \frac{m}{(\|U\|^2)^{3/2}}(U_\mu U_\nu - G_{\mu\nu}).$$

This system can be integrated easily, and we obtain

$$L(\tau, X, U) = -m(\|U\|^2)^{1/2} + U^\mu l_\mu(\tau, X) + l(\tau, X) \tag{3.13}$$

with l_μ and l arbitrary smooth functions of τ and X.

(b) In (3.13) we must change the sign of the expression under the square root.

(c) In (3.13) we must put $m = 0$.

3.4. *The functions ρ_μ and $\rho_{\mu\nu}$*

We now analyse the functions ρ_μ and $\rho_{\mu\nu}$.

(a) First we compute these functions using (3.2), (3.3) and (3.13). We get

$$\rho_\mu = U^\nu \left(\frac{\partial l_\nu}{\partial X^\mu} - (\mu \leftrightarrow \nu) \right) + \frac{\partial l}{\partial X^\mu} - \frac{\partial l_\mu}{\partial \tau}. \tag{3.14}$$

$$\rho_{\mu\nu} = \frac{1}{2} \left(\frac{\partial l_\nu}{\partial X^\mu} - (\mu \leftrightarrow \nu) \right). \tag{3.15}$$

Because ρ_μ and $\rho_{\mu\nu}$ must depend only on U, $\rho_{\mu\nu}$ is, in fact, constant and ρ_μ is linear in U:

$$\rho_\mu = 2\rho_{\mu\nu} U^\nu + c_\mu. \tag{3.16}$$

We now use (2.26). This relation gives, for $k = 0$,

$$U^\nu \frac{\partial \rho_\mu}{\partial U^\nu} = \rho_\mu \tag{3.17}$$

and, for $k = 1$,

$$U^\omega \frac{\partial \rho_{\mu\nu}}{\partial U^\omega} = 0. \tag{3.18}$$

The relation (3.18) is identically fulfilled and (3.17) yields

$$c_\mu = 0. \tag{3.19}$$

Finally, the Lorentz covariance (2.32) imposes the condition that $\rho_{\mu\nu}$ is a Lorentz invariant tensor. Because from (3.15) it follows that $\rho_{\mu\nu}$ must also be antisymmetric, we have two cases:

(i) $D \neq 2$. Then $\rho_{\mu\nu} = 0$ so the last two terms in (3.13) give a null contribution to σ_L. According to property (P3) these terms form a total divergence and so they can be discarded. It follows that L is equivalent to

$$L(\tau, X, U) = -m(\|U\|^2)^{1/2}. \tag{3.20}$$

If the Minkowski bilinear form $G_{\mu\nu}$ is given by $\mathrm{diag}(G) = (1, -1, \ldots, -1)$, then (3.20) is the usual Lagrangian for the relativistic particle in the homogeneous formulation.

(ii) $D = 2$. In this case we can have

$$\rho_{\mu\nu} = \kappa\varepsilon_{\mu\nu} \tag{3.21}$$

with $\varepsilon_{\mu\nu}$ the completely antisymmetric tensor in two dimensions. In this case we note that we can take

$$l = 0 \qquad l_\mu = -\kappa\varepsilon_{\mu\nu}X^\nu \tag{3.22}$$

and L becomes

$$L(\tau, X, U) = -m(\|U\|^2)^{1/2} + \kappa\varepsilon_{\mu\nu}X^\mu U^\nu. \tag{3.23}$$

Any other solution for l_μ and l satisfying (3.22) can only add a total divergence to the expression above.

(b) One must change the sign of the expression under the square root in (3.20) and (3.23). This case corresponds to tachionic particles.

(c) One can get a non-trivial Lagrangian only for $D = 2$, namely the last term in (3.23).

We have reobtained the results of [2, 3] rather easily.

4. The relativistic string

4.1. The relevant formulae

As in section 3, we particularize the formulae (2.14) and (2.15) for the case $p = 2$. From (2.13) it follows that one must analyse the following functions: $\sigma^{ab}_{\mu\nu}$, $\sigma^{abc}_{\mu\nu\omega}$, ρ_μ, $\rho^a_{\mu\nu}$ and $\rho^{ab}_{\mu\nu\omega}$. We get from (2.14)

$$\sigma^{ab}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 L}{\partial U^\mu{}_a \partial U^\nu{}_b} + (\mu \leftrightarrow \nu)\right) \tag{4.1}$$

$$\sigma^{abc}_{\mu\nu\omega} = \frac{1}{2}\frac{\partial}{\partial U^\mu{}_a}\left(\frac{\partial^2 L}{\partial U^\nu{}_b \partial U^\omega{}_c} - (\nu \leftrightarrow \omega)\right) \tag{4.2}$$

and from (2.15)

$$\rho_\mu = \frac{\partial L}{\partial X^\mu} - \frac{\delta L^a_\mu}{\delta \tau^a} \tag{4.3}$$

$$\rho^a_{\mu\nu} = \frac{1}{2}\left(\frac{\partial L^a_\nu}{\partial X^\mu} - \frac{\partial L^a_\mu}{\partial X^\nu} - \frac{\delta L^{ba}_{\mu\nu}}{\delta \tau^b}\right) \tag{4.4}$$

$$\rho^{ab}_{\mu\nu\omega} = \frac{1}{3}\left(\frac{\partial L^{ab}_{\nu\omega}}{\partial X^\mu} - \frac{\partial L^{ab}_{\mu\omega}}{\partial X^\nu} + \frac{\partial L^{ab}_{\mu\nu}}{\partial X^\omega}\right). \tag{4.5}$$

4.2. The functions $\sigma^{\cdots}_{\cdots}$

We first determine $\sigma^{ab}_{\mu\nu}$. From the Lorentz covariance property (2.31) and the symmetry in μ and ν (see (4.1)) it follows that $\sigma^{ab}_{\mu\nu}$ must have the following form

$$\sigma^{ab}_{\mu\nu} = U_\mu{}^1 U_\nu{}^1 f^{ab}(\|U_1\|^2), \|U_2\|^2), (U_1, U_2)) \\ + U_\mu{}^2 U_\nu{}^2 g^{ab}(\cdots) + (U_\mu{}^1 U_\nu{}^2 + U_\nu{}^1 U_\mu{}^2) h^{ab}(\cdots) + G_{\mu\nu} k^{ab}(\cdots). \tag{4.6}$$

Here f, g, h and k are smooth functions of the Lorentz invariants:

$$\zeta_1 \equiv \|U_1\|^2 \qquad \zeta^2 \equiv \|U_2\|^2 \qquad \zeta_{12} \equiv (U_1, U_2) \equiv G_{\mu\nu} U^\mu{}_1 U^\nu{}_2 . \tag{4.7}$$

Also the indices $a, b, \ldots$ are raised or lowerd with the metric δ_{ab}.

As in section 3 we first use the reparametrization invariance (2.24) and (2.25). In our case these relations are

$$U^\omega{}_c \frac{\partial \sigma^{ab}_{\mu\nu}}{\partial U^\omega{}_d} = \delta^d_c \sigma^{ab}_{\mu\nu} - \delta^a_c \sigma^{db}_{\mu\nu} - \delta^b_c \sigma^{ad}_{\mu\nu} \tag{4.8}$$

and

$$U^\mu{}_c \sigma^{ab}_{\mu\nu} = 0. \tag{4.9}$$

If we insert (4.6) into (4.9) we can establish by a straightforward computation that, in fact $\sigma^{ab}_{\mu\nu}$, must be of the form

$$\sigma^{ab}_{\mu\nu} = [(U_\mu{}^1 U_\nu{}^2 + U_\nu{}^1 U_\mu{}^2)(U_1, U_2) - U_\mu{}^1 U_\nu{}^1 \|U_2\|^2 \\ - U_\mu{}^2 U_\nu{}^2 \|U_1\|^2 - G_{\mu\nu}((U_1, U_2)^2 - \|U_1\|^2 \|U_2\|^2)] \sigma^{ab} \\ \times (\|U_1\|^2, \|U_2\|^2, (U_1, U_2)) \tag{4.10}$$

where σ^{ab} are smooth functions of the variables ζ_1, ζ_2 and ζ_{12}. Inserting (4.10) into (4.8) we will obtain a system of partial differential equations for the functions σ^{ab}. It is more convenient to use instead of the variables ζ_1, ζ_2 and ζ_{12} above, the variables ζ_1, ζ_2 and

$$\Delta \equiv (\zeta_{12})^2 - \zeta_1 \zeta_2 . \tag{4.11}$$

(This change of variables is always possible in a convenient chart.)

By some computations, one arrives at the following system

$$2\zeta_1 \frac{\partial \sigma^{ab}}{\partial \zeta_1} + 2\Delta \frac{\partial \sigma^{ab}}{\partial \Delta} = -\sigma^{ab} - \delta^a_1 \sigma^{1b} - \delta^b_1 \sigma^{a1} \tag{4.12}$$

$$2\zeta_{12} \frac{\partial \sigma^{ab}}{\partial \zeta_1} = -\delta^a_2 \sigma^{1b} - \delta^b_2 \sigma^{a1} \tag{4.13}$$

$$2\zeta_{12} \frac{\partial \sigma^{ab}}{\partial \zeta_2} = -\delta^a_1 \sigma^{2b} - \delta^b_1 \sigma^{a2} \tag{4.14}$$

$$2\zeta_2 \frac{\partial \sigma^{ab}}{\zeta_2} + 2\Delta \frac{\partial \sigma^{ab}}{\partial \Delta} = -\sigma^{ab} - \delta^a_2 \sigma^{2b} - \delta^b_2 \sigma^{a2} \tag{4.15}$$

where, in (4.13) and (4.14), one supposes that ζ_{12} is expressed as a function of ζ_1, ζ_2 and Δ (using (4.11)).

Although the preceding system looks complicated, it is remarkable that it can be solved explicitly. The computations are tedious, but do not pose difficult problems so we give only the final results. As in section 3, we have three possible choices for the domain of L.

(a) $D_+ \equiv \{(\tau, X, U) | \Delta > 0\}$. In this case we have

$$\sigma^{11}(\zeta_1, \zeta_2, \Delta) = \frac{c\zeta_2}{\Delta^{3/2}} \tag{4.16}$$

$$\sigma^{22}(\zeta_1, \zeta_2, \Delta) = \frac{c\zeta_1}{\Delta^{3/2}} \tag{4.17}$$

$$\sigma^{12}(\zeta_1, \zeta_2, \Delta) = -\frac{c\zeta_{12}}{\Delta^{3/2}} + \frac{C}{\Delta} \tag{4.18}$$

$$\sigma^{21}(\zeta_1, \zeta_2, \Delta) = -\frac{c\zeta_{12}}{\Delta^{3/2}} - \frac{C}{\Delta}. \tag{4.19}$$

Here c, C are real constants and as before, ζ_{12} is considered as a function of ζ_1, ζ_2 and Δ. But from (4.1) it is clear that we must have $\sigma^{12} = \sigma^{21}$. This fixes $C = 0$ in the formulae above. If we revert to the old variables ζ_1, ζ_2 and ζ_{12} it follows that we have

$$\sigma^{11}(\zeta_1, \zeta_2, \zeta_{12}) = \frac{c\zeta_2}{({\zeta_{12}}^2 - \zeta_1\zeta_2)^{3/2}} \tag{4.20}$$

$$\sigma^{22}(\zeta_1, \zeta_2, \zeta_{12}) = \frac{c\zeta_1}{({\zeta_{12}}^2 - \zeta_1\zeta_2)^{3/2}} \tag{4.21}$$

$$\sigma^{12}(\zeta_1, \zeta_2, \zeta_{12}) = \sigma^{21}(\zeta_1, \zeta_2, \zeta_{12}) = -\frac{c\zeta_{12}}{({\zeta_{12}}^2 - \zeta_1\zeta_2)^{3/2}}. \tag{4.22}$$

(b) $D_- \equiv \{(\tau, X, U) | \Delta < 0\}$. One must change the sign of the expression under the square root in (4.20)–(4.22).

(c) If L is defined everywhere, then necessarily

$$\sigma^{ab} = 0. \tag{4.23}$$

4.3. The Lagrangian

Now, one considers (4.1) as a system for the function L. It is not very hard to integrate this system. One obtains the following results corresponding to the cases (a)–(c) above.

(a) In this case

$$L(\tau, X, U) = c[(U_1, U_2)^2 - \|U_1\|^2\|U_2\|^2]^{1/2} + \tfrac{1}{2}\varepsilon^{ab}c_{\mu\nu}(\tau, X)U^\mu{}_a U^\nu{}_b + \mathrm{d}^a_\mu(\tau, X)U^\mu{}_a + l(\tau, X). \tag{4.24}$$

Here ε^{ab} is the completely antisymmetric symbol defined with the convention $\varepsilon^{12} = 1$, and $c_{\mu\nu}$, d^a_μ,l are smooth functions of τ and X.

(b) One changes the sign of the expression under the square root in (4.24).

(c) One takes $c = 0$ in (4.24).

4.4. The functions $\rho^{\cdots}_{\cdots}$

Instead of analysing the functions $\sigma^{abc}_{\mu\nu\omega}$ we first consider ρ_μ and $\rho^a_{\mu\nu}$.

(a) From (4.3), (4.4) and (4.24) one derives by direct computation that

$$\rho_\mu = \tfrac{1}{2}\varepsilon^{ab}c_{\mu\nu\omega}U^\nu{}_a U^\omega{}_b + \mathrm{d}^a_{\mu\nu}U^\nu{}_a + c_\mu \tag{4.25}$$

and

$$\rho^a_{\mu\nu} = \tfrac{1}{2}(\varepsilon^{ab}c_{\mu\nu\omega}U^\omega{}_b + d^a_{\mu\nu}) \tag{4.26}$$

where

$$c_{\mu\nu\omega} \equiv \frac{\partial c_{\nu\omega}}{\partial X^\mu} - \frac{\partial c_{\mu\omega}}{\partial X^\nu} - \frac{\partial c_{\nu\mu}}{\partial X^\omega} \tag{4.27}$$

$$\mathrm{d}^a_{\mu\nu} \equiv \frac{\partial \mathrm{d}^a_\nu}{\partial X^\mu} - \frac{\partial \mathrm{d}^a_\mu}{\partial X^\nu} + \varepsilon^{ab}\frac{\partial c_{\mu\nu}}{\partial \tau^b} \tag{4.28}$$

and

$$c_\mu \equiv \frac{\partial \mathrm{d}^a_\mu}{\partial \tau^a} + \frac{\partial l}{\partial X^\mu}. \tag{4.29}$$

Because the functions $\rho^{\cdots}_{\cdots}$ must depend only on U, it follows that $c_{\mu\nu\omega}$, $\mathrm{d}^a_{\mu\nu}$, c_μ must in fact be constants.

We can analyse now the consequences of (2.26). For $k = 0$ and $k = 1$ we have, respectively,

$$U^\nu{}_b \frac{\partial \rho_\mu}{\partial U^\nu{}_a} = \delta^a_b \rho_\mu \tag{4.30}$$

and

$$U^\omega{}_b \frac{\partial \rho^c_{\mu\nu}}{\partial U^\omega{}_a} = \delta^a_b \rho^c_{\mu\nu} - \delta^c_b \rho^a_{\mu\nu}. \tag{4.31}$$

Inserting (4.25) and (4.26) into (4.30) and (4.31), respectively, we easily get that we must have

$$\mathrm{d}^a_{\mu\nu} = 0 \tag{4.32}$$

$$c_\mu = 0. \tag{4.33}$$

The coefficients $c_{\mu\nu\omega}$ remain quite arbitrary, the only constraint being the complete antisymmetry, as follows from (4.27).

Finally, (2.32) is equivalent to the Lorentz invariance of the tensor $c_{\mu\nu\omega}$. So we have two distinct cases.

(i) $D \neq 3$. Then we must have

$$c_{\mu\nu\omega} = 0. \tag{4.34}$$

The equations (4.27)–(4.29) must now be considered as a system for the functions $c_{\mu\nu}$, d^a_μ and l. It is not necessary to find the most general solution. Indeed, if these functions are such that (4.27)–(4.29) are identically satisfied, then it follows that the last three terms in (4.24) give a null contribution to σ_L, i.e. they form a total derivative. So, L is equivalent to the usual Nambu–Goto Lagrangian:

$$L(\tau, X, U) = c[(U_1, U_2)^2 - \|U_1\|^2\|U_2\|^2]^{1/2}. \tag{4.35}$$

(ii) $D = 3$ In this case we can have:

$$c_{\mu\nu\omega} = \kappa\varepsilon_{\mu\nu\omega} \tag{4.36}$$

for some real κ.

Again we must consider the system (4.27)–(4.29). As above, we need only a particular solution, e.g.:

$$c_{\mu\nu} = \tfrac{1}{3}\kappa\varepsilon_{\mu\nu\omega}X^\omega \tag{4.37}$$

and d^a_μ and l can be taken to be zero. In this case (4.24) gives:

$$L(\tau, X, U) = c[(U_1, U_2)^2 - \|U_1\|^2\|U_2\|^2]^{1/2} + \tfrac{1}{6}\kappa\varepsilon_{\mu\nu\omega}\varepsilon^{ab}U^\mu{}_a U^\nu{}_b X^\omega. \tag{4.38}$$

Any other solution of (4.27)–(4.29) only adds a total derivative to this expression.

One may ask if it is still necessary to analyse the conditions on the functions $\sigma^{abc}_{\mu\nu\omega}$ and $\rho^{ab}_{\mu\nu\omega}$. That it is not the case can be seen as follows. Both expressions (4.35) and (4.38) can be easily seen to verify the following relations

$$L \circ \dot{\phi}_\xi = (\det(A))L \tag{4.39}$$

for any $\xi \in (\mathrm{Diff}(P))_0$, and

$$L \circ \dot{\phi}_{L,0} = L \tag{4.40}$$

for any proper orthochronous Lorentz transformation L. Also $\dot{\phi}_{1,a}$ changes L by a total divergence.

These facts easily imply that (2.20) and (2.30) are identically satisfied, so indeed (4.35) and (4.38) are the most general solution of these invariance conditions, up to a total divergence.

(b) One changes the sign of the expression under the square root in (4.35) and (4.38).

(c) In this case we can have non-trivial Lagrangian defined everywhere *iff* $D = 3$, namely (4.38) with $c = 0$.

5. Remarks

(i) The Lagrangian (4.35) appearing in case (a) corresponds to the usual relativistic string submitted to the condition that every point of it is moving with a subluminal velocity. The case (b) obviously corresponds to a 'tachionic' string for which all points are travelling with supraluminal velocities.

(ii) It is interesting to note that the term of a Chern–Simons nature appearing in (4.38) verifies (4.39) for any $\xi \in \mathrm{Diff}(P)$. This property is not shared by the usual Nambu–Goto Lagrangian (4.35).

(iii) One could wonder if one could have obtained the same result using, instead of the invariance condition (2.7), the more restrictive condition

$$\Phi^{*}\theta_L = \theta_L .$$

However, a condition of this type for $\Phi = \dot{\phi}$ is equivalent to the strict invariance of the action functional with respect to ϕ. The condition (2.7) is more general in the sense that ϕ can leave the action functional invariant, up to a trivial action. But, as remarked above, the Chern–Simons term is invariant with respect to translations $\phi_{1,a}$ only up to a total divergence. So the more restrictive invariance condition above rules out this contribution.

6. Conclusions

We have succeeded in deriving the Nambu–Goto Lagrangian only from invariance considerations. The proof is highly non-trivial; as always this shows that the starting point i.e. the symmetry considerations have a deep physical meaning.

An expression of a Chern–Simons character appears in a three-dimensional Minkowski space. Let us mention that the physical implications of a contribution of this type have been recently analysed in the literature [11].

It also seems desirable to extend the analysis to the general case of a p-brane. This has been done recently by a different computational method.

It is clear that the formalism used above is powerful enough to analyse the case of two or more interacting objects, but the computations will be very complicated. Nevertheless, it is easy to show that, in this formalism, one can have non-trivial interactions, so one is able to circumvent the well-known 'no-interaction' theorems (see [10] and references cited therein).

Another interesting problem is that it is well known that the Poliakov action is usually preferred to the Nambu–Goto action. The method used in this paper is capable of dealing with this case also. However, one has to include new independent variables in the Lagrangian (namely the metrics of the parameter manifold) and to enlarge appropriately the group of Noetherian symmetries (including the Weyl scaling invariance also). It is expected that we will essentially obtain the Poliakov action in this way. The computations will be much more complicated.

Finally we stress that our results depend essentially on the very particular expression of the Poincaré–Cartan form (2.3) which ensures the key properties (P3) and (P4). Let us note that the property (P2) depends only on the first two terms in (2.3). This explains why one finds other Poincaré–Cartan forms in the literature (e.g. [12]). However, as remarked in [6], if one cuts the sum in (2.3) at a value strictly smaller than p, then the group of transformations verifying (2.7) is strictly smaller than the whole group of Noetherian symmetries. One gets the whole group only for the whole sum. So, we can ask what results we would have obtained if instead of (2.3), we have used the Poincaré–Cartan form from [12]. In the case $p = 1$ the two forms coincide so there is no difference, but in the case $p = 2$, if one considers in (2.3) only the first two terms and imposes (2.20) and (2.30), one loses exactly the Nambu–Goto term.

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References

[1] Scherk J 1975 *Rev. Mod. Phys.* **47** 123
[2] Lévy-Leblond J L 1969 *Commun. Math. Phys.* **12** 67
[3] Grigore D R 1991 Generalized Lagrangian dynamics and Noetherian symmetries *Int. J. Mod. Phys.* A (CERN-TH-6170/91) to appear
[4] Souriau J M 1970 *Structure des Systemes Dynamiques* (Paris: Dunod)
[5] Betounes D 1984 *Phys. Rev.* D **29** 599
[6] Betounes D 1987 *J. Math. Phys.* **28** 2347
[7] Rund H 1985 *Lecture Notes in Pure and Applied Mathematics* **100** 155
[8] Grigore D R and Popp O T 1992 *Ann. Phys.* at press
[9] Grigore D R 1992 Gauge theories in the Souriau–Lagrange framework for classical field theories *Preprint* CERN-TH-6039/91
[10] Kuntzle H P 1974 *Symposia Math.* **14** 53
[11] Zaikov R P 1991 *Mod. Phys. Lett.* A **6** 1453
[12] Goldschmidt H and Sternberg S 1973 *Ann. Inst. Fourier* **23** 203